Tutor-marked Assignment

M828 TUTOR-MARKED ASSIGNMENTS 1987

Please ensure that your TMAs reach your tutor by the following dates:

TMA 01	17 APRIL 1987
TMA 02	5 JUNE 1987
TMA 03	24 JULY 1987
TMA 04	25 SEPTEMBER 1987.

Each TMA should be accompanied by a completed PT3 form. (PT3 forms will be sent to you with the assignment questions.) Please complete Section 1 of the form in as much detail as possible. It is essential that you complete the following:

Name
Address
Date sent to tutor
Personal Identifier
Course and Assignment Number.

M828 TUTOR-MARKED ASSIGNMENT 01

Please make sure that the assignment number is correctly entered on your PT3 form as

M828 01.

THE FINAL DATE FOR SUBMISSION OF THIS ASSIGNMENT IS 17 APRIL 1987.

1. Show that $w = w(z) = \frac{z-1}{z+1}$ maps $D_0 = \{z\colon \operatorname{Im} z > 0,\ |z| > 1\}$ 1-1 and conformally onto $D_1 = \{w\colon \operatorname{Re} w > 0,\ \operatorname{Im} w > 0\}$, i.e. the first quadrant in the w-plane; and hence that $\zeta = \zeta(z) = \left(\frac{z-1}{z+1}\right)^2$ maps D_0 onto $D_2 = \{\zeta : \operatorname{Im} \zeta > 0\}$, i.e. the upper half of the ζ-plane.

   Consider the flow in the ζ-plane given by the stream function $\psi(\zeta, \eta) = -\eta$, where $\zeta = \xi + i\eta$. Determine the stream function of the corresponding flow in the z-plane.

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2. Reflecting D_0 in the x-axis, then apart from the parts of the x-axis outside the segment $[-1,1]$, we obtain the z-plane with the unit disc removed. The mapping $Z = \frac{1}{2}(z + \frac{1}{z})$ maps $\{|z| > 1\}$ onto the Z-plane with the segment $[-1,1]$ of the X-axis removed, where $Z = X + iY$.

Find the stream function in the z-plane corresponding to the stream function $\Psi(X,Y) = -Y$ in the Z-plane and indicate why we get a flow in D_0. Show that this flow is different from that found in 1.

Why is the flow in D_0 found in 2 regarded as that obtained by inserting a rigid semi-cylinder into a uniform laminar flow in a half-space?

3. Let D be the semi-disc given by $\{|z| < 1\} \cap \{\text{Im } z > 0\}$, i.e. that part of the unit disc that lies in the upper half-plane. Let $u(x,y)$ be the harmonic function in D that is α on the upper semi-circle and β on the segment $(-1,1)$ of the x-axis. By considering the value of $\arg \frac{z-1}{z+1}$ on these two parts of the boundary ∂D of D, find $u(x,y)$.

4. Show that $\zeta = (\frac{1+z}{1-z})^2$ maps D onto H, the upper half of the ζ-plane with the circular part of ∂D corresponding to the negative ξ-axis and the straight part of ∂D corresponding to the positive ξ-axis, where $\zeta = \xi + i\eta$. Using Poisson's formula for a half-plane (Dettman Ex.6.3, #2) find $v(\xi,\eta)$ the harmonic function in H with

$$v(\xi,0) = \begin{cases} \alpha & (\xi < 0), \\ \beta & (\xi > 0). \end{cases}$$

Via the conformal mapping $z \mapsto \zeta$, find again the function $u(x,y)$ of 3.

5. Note that the forms of $u(x,y)$ in 3 and 4 are quite different. Try and show that they do, in fact, represent the same function. Do not spend too much time on this since it is really rather tricky. However, it does illustrate that even in rather simple-looking situations, the same function can take different, not easily reconcilable forms.

Tutor-marked Assignment

M828 TUTOR-MARKED ASSIGNMENT 02

Please send your answers to the complete assignment to your tutor together with a completed PT3 form. Make sure that the assignment number is correctly entered as

M828 02

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THE FINAL DATE FOR SUBMISSION OF THIS ASSIGNMENT IS

5 JUNE 1987

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1. Find the exact solution of the initial value problem

$$\dot{x}(t) = 2t(1 + x), \quad x(0) = 0.$$

Starting with $x_0(t) \equiv 0$, calculate $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ as in the proof of Theorem 7.2-1, and compare these results with the exact solution.

2. Consider

$$w'' + \frac{1}{z^2} w' - \frac{1}{z^3} w = 0.$$

(i) Show that $z = 0$ is an irregular singular point.

(ii) Note that $w_1(z) = z$ is a solution and hence find a second, independent, solution $w_2(z)$.

(iii) Show that $w_2(z)$ could not have been obtained by assuming solutions of the form $w(z) = z^\alpha \sum_0^\infty a_k z^k$.

3. Let $f(x)$ be a real-valued continuous function on $[-1,1]$. Define a_k $(k=0,1,\dots)$ by

$$\frac{2}{2k+1} a_k = \int_{-1}^{1} f(x)\, P_k(x)\, dx,$$

where $P_k(k)$ is the kth Legendre polynomial. For any positive integer n and a real-valued polynomial $p(x)$ of degree n, show that

$$\int_{-1}^{1} \left|f(x) - \sum_{0}^{n} a_k P_k(x)\right|^2 dx \le \int_{-1}^{1} |f(x) - p(x)|^2 dx.$$

4. Show that if $t_0 > 0$ is small enough, then the iterative method used to prove Theorem 7.2-1 gives a solution of

$$x(t) = \int_0^t (t-\tau)^2 e^{-x(\tau)} d\tau \qquad (0 \le t \le t_0).$$

This means that if $x_0(t) \equiv 0$,

$$x_1(t) = \int_0^t (t-\tau)^2 d\tau,$$

$$x_2(t) = \int_0^t (t-\tau)^2 e^{-x_1(\tau)} d\tau,$$

and so on, then $x_n(t) \to x(t)$ $(n \to \infty)$ for $0 \le t \le t_0$ and $x(t)$ is a solution of the integral equation.

Tutor-marked Assignment

M828 TUTOR-MARKED ASSIGNMENT 03

Please send your answers to the complete assignment to your tutor together with a completed PT3 form. Make sure that the assignment number is correctly entered as

M828 03.

THE FINAL DATE FOR SUBMISSION OF THIS ASSIGNMENT IS

24 JULY 1987.

1. Prove that for any real number u such that $\sin \frac{1}{2} u \neq 0$,

$$\sin \frac{1}{2} u + \sin \frac{3}{2} u + \ldots + \sin(n + \frac{1}{2})u = \frac{\left[\sin \frac{n+1}{2} u\right]^2}{\sin \frac{1}{2} u},$$

by considering

$$\sin(k + \frac{1}{2})u = \mathrm{Im}[e^{i(k+\frac{1}{2})u}].$$

Let f(t) be integrable over $[0, 2\pi]$ and of period 2π. If S_k is defined as on p.353 of Dettman, show that

$$S_0 + \ldots + S_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u+t) \left[\frac{\sin \frac{n+1}{2} u}{\sin \frac{1}{2} u}\right]^2 du.$$

By taking $f(t) \equiv 1$, deduce that

$$n + 1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{\sin \frac{n+1}{2} u}{\sin \frac{1}{2} u}\right]^2 du,$$

and hence that

$$\sigma_n - f(t) = \frac{S_0+\ldots+S_n}{n+1} - f(t) = \frac{1}{2\pi(n+1)} \int_{-\pi}^{\pi} [f(u+t) - f(t)] \left[\frac{\sin \frac{n+1}{2} u}{\sin \frac{1}{2} u}\right]^2 du.$$

Hence show that if f(t) is continuous at t_0 then

$$\sigma_n(t_0) \to f(t_0) \quad (n \to \infty).$$

What conclusion can you draw when we assume that f(t) is continuous for all t?

2. Suppose that f(t) is continuous for $-\infty < t < \infty$ and that $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges. Show that for x real and non-zero,

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-ix\tau}\, d\tau$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau - \tfrac{\pi}{x}) e^{-ix\tau}\, d\tau,$$

and deduce that $G(x) \to 0 \quad (|x| \to \infty)$.

Give an alternative proof of this result using Theorem 8.1.1 on p.350 of Dettman.

3. Solve the integrodifferential equation

$$\frac{dy}{dt} + 2\int_0^t y(\tau)\, d\tau = e^{-2t}$$

for $t > 0$ subject to $\lim_{t\to 0+} y(t) = y_0$.

Remark: In the solution to P.18 on p.75 of the notes, I have in the first part dealt with the case $y_0 = 0$ and in the second part adjusted the solution to fit the general initial value condition. One can deal with the general case directly by noting that

$$\int_0^{\infty} \frac{dy}{d\tau} e^{-iz\tau}\, d\tau = y(\tau)e^{-iz\tau}\Big|_0^{\infty} + iz\int_0^{\infty} y(\tau)e^{-iz\tau}\, d\tau$$

$$= -y_0 + \sqrt{2\pi}\, iz\, F[y].$$

4. Solve the integral equation

$$u(t) = 1 + \int_0^t (t - \tau)^3 u(\tau)\, d\tau \qquad (t \geq 0).$$

Tutor-marked Assignment

M828 TUTOR-MARKED ASSIGNMENT 04

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M828 04

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THE FINAL DATE FOR SUBMISSION OF THIS ASSIGNMENT IS

25 SEPTEMBER 1987

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1. Find the asymptotic expansion of

$$\int_0^\infty \frac{e^{-zt}}{1+t^4}\,dt$$

Th. 10.3.1 p 446

as $z \to \infty$ in $\{z: \frac{-\pi}{2} + \alpha \le \arg z \le \frac{\pi}{2} - \alpha\}$, $0 < \alpha < \frac{\pi}{2}$.

2. Find the asymptotic expansion of

$$\int_0^1 e^{-xt^2}\, t^{-1/3} \cos t \, dt$$

as $x \to \infty$ through real values.

3. The integral $\int (s\, e^{1-s})^z ds$ on p.447 of Dettman can be considered as $\int e^{zh(\zeta)}$ as on p.453 with $h(\zeta) = 1 - \zeta + \log \zeta$.

Show that in this particular case the steepest descent method as described on p.453 leads precisely to the argument on pp.448-9.

[Remark: This is quite a tricky problem.]

4. Assuming that the method of stationary phase is applicable show that

$$f(x) = \int_0^{\infty} \cos[x(t^3-t)]dt \sim \left(\frac{\pi}{3xt_0}\right)^{1/2} \cos[x\ (t_0^3-t_0) + \frac{\pi}{4}] \qquad (t_0 = \frac{1}{\sqrt{3}})$$

as x → ∞ through real values.

How would you go about showing that this result is really valid?

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